Archimedes' famous-theorem

Henri Gouin

Archimedes' famous-theorem

Henri Gouin†
Aix-Marseille Université, CNRS, Centrale Marseille, M2P2 UMR 7340
13451 Marseille France
E-mails: henri.gouin@univ-amu.fr; henri.gouin@yahoo.fr

(Received 22 October 2015)

In his treatise addressed to Dositheus of Pelusium (Calinger (1999)), Archimedes of Syracuse obtained the result of which he was the most proud: *a sphere has two-thirds the volume of its circumscribing cylinder* (see Fig. 1). At his request a sculpted sphere and cylinder were placed on his tomb near Syracuse (Rorres (2011)).
Usually, it is admitted that to find this formula, Archimedes used a half polygon inscribed in a semicircle; then he performed rotations of these two figures to obtain a set of trunks in a sphere. This set of trunks allowed him to determine the volume.
In our opinion, Archimedes was so clever that he found the proof with shorter demonstration. Archimedes did not need to know π to prove the result and the Pythagorean theorem was probably the key to the proof.

Keywords: Archimedes, science history
MSC numbers: 01A20, 03-03

Proof

See Fig. 2a:

The area generated by the rotation of segment AC around $\overrightarrow{Oz}$ at level ℓ is proportional to the square of radius r (the proportionality coefficient is π).

The area generated by the rotation of segment AB around $\overrightarrow{Oz}$ at level ℓ is proportional to the radius square of a radius which is, on the basis on the Pythagorean theorem, $\sqrt{r^2 - \ell^2}$.

The area generated by the rotation of segment BC around $\overrightarrow{Oz}$ at level ℓ is proportional to $r^2 - (r^2 - \ell^2) = \ell^2$.

See Fig. 2b:

Volume V_1 of the domain included between the cylinder and the sphere is therefore twice the volume of a cone whose the base is that of the cylinder and the height is the radius of the sphere.
(Archimedes knew the volume of the cone which is the extension of the volume of a pyramid with n faces, $(3 \leq n \in N)$).

Consequence:

Two times the volume of the cone is $V_1 = (2/3)\, r\, S$ where S is the surface of the cone basis.

The volume of the cylinder is $V_2 = 2r\, S$.

The volume of the sphere is $V_3 = V_2 - V_1$

† dedicato ai miei amici italiani

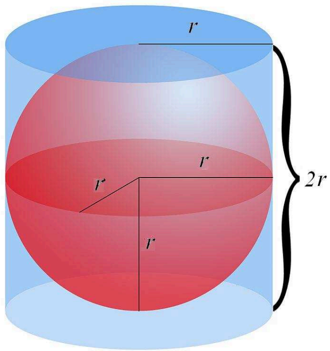

Figure 1. *A sphere has 2/3 the volume of its circumscribing cylinder. The radius of the sphere is r.*

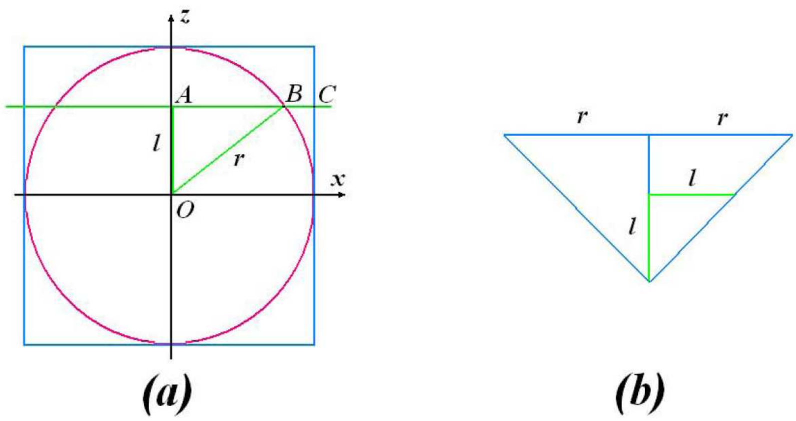

Figure 2. Sphere, cylinder and cone.

Conclusion:

$$V_3 = \frac{2}{3}\, V_2$$

□

Remark: If we know - as Archimedes did (McKeeman (2012)) - the signification of π,

$$V_2 = 2r\,(\pi r^2) \quad \text{and consequently} \quad V_3 = \frac{4}{3}\,\pi r^3\,.$$

References

Calinger, Ronald. "A Contextual History of Mathematics" p. 150. Prentice-Hall, ISBN 0-02-318285-7. *Shortly after Euclid, compiler of the definitive textbook, came Archimedes of Syracuse* (ca. 287 – 212 BC), *the most original and profound mathematician of antiquity* (1999).

Rorres, Chris. "Tomb of Archimedes Illustrations". Courant Institute of Mathematical Sciences (2011).

McKeeman, Bill. "The Computation of Pi by Archimedes". Matlab Central (2012).